

A GEOMETRIC REALIZATION OF DEGENERATE PRINCIPAL SERIES REPRESENTATIONS OF SYMPLECTIC GROUPS

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ABSTRACT. The multidimensional quantization procedure, proposed by the first author and its modifications (reduction to radicals and lifting on $U(1)$ -coverings) give us a almost universal theoretical tools to find irreducible representations of Lie groups. By using this method and the root theory, we realize in this paper the representations of the degenerate principal series of symplectic groups.

1. INTRODUCTION

Let us consider a Lie group G and its Lie algebra \mathfrak{g} . The group G acts on its Lie algebra \mathfrak{g} by adjoint representation Ad and in the dual vector space \mathfrak{g}^* by coadjoint action $K := \text{coAd}$. The vector space \mathfrak{g}^* is therefore divided into the disjoint union of coadjoint orbits (or K -orbits). Each coadjoint orbit $\Omega \in \mathfrak{g}^*/G$ admits a natural G -homogeneous symplectic structure, corresponding to the Kirillov form B_Ω associated with the bilinear form

$$B_F(X, Y) := \langle F, [X, Y] \rangle,$$

where $F \in \mathfrak{g}^*$, the kernel $\ker B_F$ of which is just isomorphic to the Lie algebra $\mathfrak{g}_F := \text{Lie } G_F$ of the stabilizer G_F of a fixed point $F \in \Omega$. Therefore, the triple (Ω, B_Ω, G) is a homogeneous symplectic manifold (Hamiltonian system) with a *flat action*, i.e.

$$\{f_X, f_Y\} = f_{[X, Y]}, \forall X, Y \in \mathfrak{g},$$

where f_X is such a function that $df_X = -\iota(\xi_X)B_\Omega$ and

$$\xi_X(m) := \left. \frac{d}{dt} \right|_{t=0} \exp(tX)m,$$

see [1] of G . Following the well known classification theorem of A. Kirillov-B. Kostant-Souriaux, every homogeneous symplectic manifold with a flat action of G is locally diffeomorphic to a coadjoint orbit of G or its central extension \tilde{G} by \mathbb{R} .

Date: September 15, 1998 and, in revised form, ..., 1998.

1991 Mathematics Subject Classification. Primary 22E45; Secondary 46E25, 20C20.

Key words and phrases. Symplectic geometry, Lie group representation, orbit method.

The first author was supported in part by a generous Grant from The IMU Exchange Commission on Development and the Abdus Salam ICTP.

This means that all the Hamiltonian systems with a flat action of G are locally classified by the coadjoint orbits of G or its central extension \tilde{G} by \mathbb{R} . The Hamiltonian systems can be quantized to become quantum systems with unitary symmetry representations of G . In order to do this, one uses affine connection ∇ with the symplectic curvature

$$\text{Curv}(\nabla) = \frac{2\pi i}{h} \omega$$

on the homogeneous symplectic manifold (M, ω, G) with a flat action of G to produce a *quantization procedure*

$$Q : C^\infty(\Omega) \rightarrow \mathcal{L}(\mathbf{H}),$$

$$f \mapsto Q(f) := f + \frac{h}{2\pi i} \nabla_{\xi_f},$$

where ξ_f is the symplectic gradient of $f \in C^\infty(\Omega)$, i.e.

$$\iota(\xi_f)\omega = -df$$

and \mathbf{H} is a separable Hilbert space. This correspondence is a geometric quantization procedure because it satisfies the following relations

$$Q(\{f, g\}) = \frac{2\pi i}{h} [Q(f), Q(g)],$$

$$Q(1) = \text{Id}_{\mathbf{H}}.$$

It means that this correspondence defines a homomorphism

$$\Lambda : C^\infty(\Omega) \rightarrow \mathcal{L}(\mathbf{H}),$$

$$f \mapsto \Lambda(f) := \frac{2\pi i}{h} Q(f),$$

i.e. a representation of the Lie algebra of smooth functions with respect to the Poisson brackets in the Hilbert space \mathbf{H} .

An element $X \in \mathfrak{g}$ can be considered as a function on \mathfrak{g}^* and therefore the restriction $X|_\Omega$ belongs to $C^\infty(\Omega)$ and we can obtain the corresponding Hamiltonian field X_Ω from the condition

$$\iota(X_\Omega)B_\Omega = dX|_\Omega.$$

The condition asserting that the action is flat means that the correspondence

$$X \in \mathfrak{g} \mapsto X|_\Omega \in C^\infty(\Omega)$$

is the Lie algebra homomorphism and we have a representation of Lie algebra \mathfrak{g}

$$X \in \mathfrak{g} \mapsto \Lambda(X) := \frac{2\pi i}{h} Q(X|_\Omega)$$

by auto-adjoint operators in the Hilbert space \mathbf{H} , which is as usually constructed as the completion of some subspace of so called partially invariant partially holomorphic sections of the quantum bundle, associated with some fixed polarization. This is what we means the *multidimensional quantization procedure*.

One considers also the *reductions* of this multidimensional quantization procedure in the following sense. Let us denote the (solvable) radical of G_F by R_F and the unipotent radical by ${}^u R_F$.

In the above construction, there appeared some so called *Mackey obstruction*. In order to kill this Mackey obstruction, we supposed some additional conditions on the action of stabilizers G_F on the dual object of inducing subgroups \widehat{H}_0 is trivial, see [1]. Duflo [4] proposed another method of Killing this Mackey obstruction by *lifting to the $\mathbf{Z}/(2)$ coverings* of the stabilizers. One has a lifting of any homomorphism of G_F into the symplectic groups of the orbit at the fixed point F onto a homomorphism of $\mathbf{Z}/(2)$ covering $G_F^{\mathbf{Z}/(2)}$ of G_F into the metaplectic group $\text{Mp}(T_F\Omega)$, following the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{Z}/(2) & \longrightarrow & G_F^{\mathbf{Z}/(2)} & \longrightarrow & G_F \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{Z}/(2) & \longrightarrow & \text{Mp}(T_F\Omega) & \longrightarrow & \text{Sp}(T_F\Omega) \longrightarrow 1 \end{array}$$

Replacing the metaplectic groups Mp by the (complex) metaplectic Mp^c groups, Tran Vui [9] and Tran Dao Dong [5] considered the same *lifting to $U(1)$ coverings* of the the stabilizers. One has to lift each homomorphism of G_F into the symplectic groups of the orbit at the fixed point F onto a homomorphism of $U(1)$ covering $G_F^{U(1)}$ of G_F into the metaplectic group $\text{Mp}^c(T_F\Omega)$, following the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & U(1) & \longrightarrow & G_F^{U(1)} & \longrightarrow & G_F \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & U(1) & \longrightarrow & \text{Mp}^c(T_F\Omega) & \longrightarrow & \text{Sp}(T_F\Omega) \longrightarrow 1 \end{array}$$

It was shown that these modifications give us some privileges for constructing irreducible unitary representations. The theory was settled in the general context, but still it is difficult to realize in concrete situations. The *discrete series representations* of semi-simple Lie groups are realized by these constructions as globalization of Harish-Chandra modules, see [7],[10].

The *degenerate principal series representations* of semisimple Lie groups were constructed and studied in many works, beginning from Harish-Chandra, updated by Vogan and others see e.g. Vogan [8] for the general semisimple Lie groups, Lee [6] for symplectic groups, etc. The theories however were purely analytic. It is natural to try to use the developed geometric quantization method to describe these

representation. In this paper, the general theory of geometric quantization is applied in the situation of symplectic groups and we describe the computation results.

2. STRUCTURE OF COADJOINT ORBITS

Let $G = \mathrm{Sp}_{2n}(\mathbb{R})$ be the symplectic group, $\mathfrak{g} := \mathrm{Lie} G$ its Lie algebra and $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$ the vector space dual to the Lie algebra \mathfrak{g} . We study in detail the coadjoint orbits of the symplectic group.

Lemma 2.1. *The Lie algebra \mathfrak{g} , its dual vector space \mathfrak{g}^* are realized by matrices and the coadjoint action of G in \mathfrak{g}^* is just the conjugation*

$$\begin{aligned} K = \mathrm{coAd} : G \times \mathfrak{g}^* &\rightarrow \mathfrak{g}^*, \\ (g, F) &\mapsto K(g)F = gFg^{-1}. \end{aligned}$$

Proof. Recall that the adjoint action of G on \mathfrak{g} can be realized as conjugation

$$\begin{aligned} \mathrm{Ad} : G \times \mathfrak{g} &\rightarrow \mathfrak{g}, \\ (2.1) \quad (g, X) &\mapsto \mathrm{Ad}(g)X = gXg^{-1}. \end{aligned}$$

Let us denote the trace of a matrix by tr . To each matrix $Y \in \mathfrak{g}$ we define the associate functional $F_Y \in \mathfrak{g}^*$ by

$$(2.2) \quad \langle F_Y, X \rangle = \mathrm{tr}(Y.X), \forall X \in \mathfrak{g}.$$

Then the map $\mathfrak{g} \rightarrow \mathfrak{g}^*$, $Y \mapsto F_Y$ is an isomorphism from \mathfrak{g} onto \mathfrak{g}^* . We identify therefore $F \in \mathfrak{g}^*$ with a matrix denote by the same letter F and 2.2 become

$$(2.3) \quad \langle F, X \rangle = \mathrm{tr}(F.X)$$

Recall that

$$\langle K(g)F, X \rangle = \langle F, \mathrm{Ad}(g^{-1})X \rangle$$

and using 2.1 and 2.3 we have

$$\begin{aligned} \langle K(g)F, X \rangle &= \langle F, \mathrm{Ad}(g^{-1})X \rangle \\ &= \langle F, g^{-1}Xg \rangle \\ &= \mathrm{tr}(F.g^{-1}Xg) \\ &= \mathrm{tr}(gFg^{-1}X) \\ &= \langle gFg^{-1}, X \rangle, \end{aligned}$$

for all $g \in G$, $x \in \mathfrak{g}$ and $F \in \mathfrak{g}^*$.

We have therefore,

$$K(g)F = gFg^{-1}, \forall g \in G, \forall F \in \mathfrak{g}^*.$$

□

Remark 2.2. We fix a special element $F \in \mathfrak{g}^*$ presented by a matrix of type

$$F = \begin{pmatrix} \begin{bmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{bmatrix} & & & & 0 \\ & \ddots & & & \\ & & \begin{bmatrix} 0 & -\lambda_r \\ \lambda_r & 0 \end{bmatrix} & & \\ & & & \begin{bmatrix} 0 & \dots \\ \vdots & \dots \end{bmatrix} \\ 0 & & & & \end{pmatrix},$$

where $\lambda_i \in \mathbb{R}$, $\lambda_i > 0$, $\lambda_i \neq \lambda_j$, $i, j = 1, \dots, n$. Consider the coadjoint orbit Ω_F passing through this point F ,

$$\Omega_F = \{K(g)F \mid g \in G\}.$$

Proposition 2.3. *The stabilizer G_F of F consists of the matrices of type*

$$\begin{pmatrix} g_{11} & & & & \\ & g_{22} & & & \\ & & \dots & & \\ & & & g_{rr} & \\ & & & & g_{r+1,r+1} \end{pmatrix},$$

where

$$g_{ii} = \begin{bmatrix} \cos \lambda_i & \sin \lambda_i \\ \sin \lambda_i & \cos \lambda_i \end{bmatrix},$$

$i = 1, \dots, r$, $g_{r+1,r+1} \in \text{Sp}_{2(n-r)}(\mathbb{R})$.

Proof. We can write each element of $\text{Sp}_{2n}(\mathbb{R})$ into block format

$$g = \begin{pmatrix} g_{11} & g_{12} & \dots & g_{1r} & g_{1,r+1} \\ g_{21} & g_{22} & \dots & g_{2r} & g_{2,r+1} \\ \dots & \dots & \dots & \dots & \dots \\ g_{r+1,1} & g_{r+1,2} & \dots & g_{r+1,r} & g_{r+1,r+1} \end{pmatrix},$$

where g_{ij} is a 2×2 matrix, for $i, j = 1, \dots, r$, $g_{i,r+1}$ is a $2 \times 2(n-r)$ matrix, for $i = 1, \dots, r$, $g_{r+1,j}$ is a $2(n-r) \times 2$ matrix, for $j = 1, \dots, r$, $g_{r+1,r+1}$ is a $2(n-r) \times 2(n-r)$ matrix. We write F in the same format form

$$g = \begin{pmatrix} F_{11} & F_{12} & \dots & F_{1r} & F_{1,r+1} \\ F_{21} & F_{22} & \dots & F_{2r} & F_{2,r+1} \\ \dots & \dots & \dots & \dots & \dots \\ F_{r+1,1} & F_{r+1,2} & \dots & F_{r+1,r} & F_{r+1,r+1} \end{pmatrix},$$

where $F_{ij} = 0$ is 2×2 matrix, for $i, j = 1, \dots, r$, $i \neq j$, $F_{ii} = \begin{bmatrix} 0 & -\lambda_i \\ \lambda_i & 0 \end{bmatrix}$, for $i = 1, \dots, r$, $F_{i,r+1} = 0$ is $2 \times 2(n-r)$ matrix, for $i = 1, \dots, r$, $F_{r+1,j} = 0$ is

$2(n-r) \times 2$ matrix, for $j = 1, \dots, r$, $F_{r+1, r+1} = 0$ is $2(n-r) \times 2(n-r)$ matrix. We recall that

$$\begin{aligned} G_F &= \{g \in G \mid K(g)F = F\} \\ &= \{g \in G \mid gFg^{-1} = F\} = \{g \in G \mid gF = Fg\}. \end{aligned}$$

We deduce from $gF = Fg$ that $g \in G_F$ if and only iff the following 3 conditions hold:

$$(2.4) \quad g_{r+1, j} F_{jj} = 0, \forall j = 1, \dots, r,$$

$$(2.5) \quad F_{ii} g_{i, r+1} = 0, \forall i = 1, \dots, r,$$

$$(2.6) \quad g_{ij} F_{jj} = F_{ii} g_{ij}, \forall i, j = 1, \dots, r.$$

Because $F_{ii} = \begin{bmatrix} 0 & -\lambda_i \\ \lambda_i & 0 \end{bmatrix}$, for all $i = 1, \dots, r$ are invertible then from 2.4 and 2.5 we have

$$(2.7) \quad g_{r+1, j} = 0, \forall j = 1, \dots, r,$$

$$(2.8) \quad g_{i, r+1} = 0, \forall i = 1, \dots, r.$$

We now resolve the equation 2.8. Let us denote

$$g_{ij} = \begin{bmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{bmatrix}.$$

Then

$$\begin{aligned} g_{ij} F_{jj} &= \lambda_j \begin{bmatrix} b_{ij} & -a_{ij} \\ d_{ij} & -c_{ij} \end{bmatrix}, \\ F_{ii} g_{ij} &= \lambda_i \begin{bmatrix} -c_{ij} & -d_{ij} \\ a_{ij} & b_{ij} \end{bmatrix}. \end{aligned}$$

From the condition

$$g_{ij} F_{jj} = F_{ii} g_{ij}$$

we deduce that

$$\begin{cases} \lambda_i b_{ij} - \lambda_j c_{ij} = 0 \\ \lambda_j b_{ij} + \lambda_i c_{ij} = 0 \\ \lambda_i a_{ij} - \lambda_j d_{ij} = 0 \\ \lambda_j a_{ij} - \lambda_i d_{ij} = 0 \end{cases}$$

what is equivalent that

$$a_{ij} = b_{ij} = c_{ij} = d_{ij} = 0,$$

because of the assumption, $\lambda_i \neq \lambda_j$; $\lambda_i, \lambda_j > 0$, i.e.

$$(2.9) \quad g_{ij} = 0, \forall i, j = 1, \dots, r, i \neq j.$$

From 2.7-2.9 we conclude that the element g , as matrix should be of the diagonal form

$$g = \text{diag}(g_{11}, g_{22}, \dots, g_{rr}, g_{r+1, r+1}),$$

where g_{ii} is a 2×2 matrix, for all $i = 1, \dots, r$, $g_{r+1, r+1}$ is a $2(n-r) \times 2(n-r)$ matrix.

Denote

$$g_{i1} = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}, \quad F_i = \begin{bmatrix} 0 & -\lambda_i \\ \lambda_i & 0 \end{bmatrix}$$

we have

$$g_{i1}F_i = \lambda_i \begin{bmatrix} b_i & -a_i \\ d_i & -c_i \end{bmatrix},$$

$$F_i g_{i1} = \lambda_i \begin{bmatrix} -c_i & -d_i \\ a_i & b_i \end{bmatrix}.$$

Our condition means that

$$\lambda_i \begin{bmatrix} b_i & -a_i \\ d_i & -c_i \end{bmatrix} = \lambda_i \begin{bmatrix} -c_i & -d_i \\ a_i & b_i \end{bmatrix},$$

what is equivalent to the conditions

$$(2.10) \quad \begin{cases} a_i &= d_i, \\ b_i &= -c_i. \end{cases}$$

Let us denote

$$J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

then the matrix of the symplectic form is

$$J_n = \begin{pmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & & 0 \\ & \ddots & & \\ & & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \\ 0 & & & [J_{n-r}] \end{pmatrix} = \begin{pmatrix} [J_1] & & & 0 \\ & \ddots & & \\ & & [J_1] & \\ & & & [J_{n-r}] \end{pmatrix}.$$

We have therefore

$$gJ_ng^t = \begin{pmatrix} [g_{11}J_1g_{11}^t] & & & 0 \\ & \ddots & & \\ & & [g_{rr}J_1g_{rr}^t] & \\ & & & [g_{r+1, r+1}J_{n-r}g_{r+1, r+1}^t] \end{pmatrix},$$

and the condition

$$gJ_ng^t = J_n,$$

guaranting that $g \in \mathrm{Sp}_{2n}(\mathbb{R})$ is equivalent to the conditions

$$\begin{cases} g_{ii} J_1 g_{ii}^t &= J_1, \forall i = 1, \dots, n, \\ g_{r+1, r+1} J_{n-r} g_{r+1, r+1}^t &= J_{n-r} \end{cases}.$$

It means also that $g_{ii} \in \mathrm{Sp}_2(\mathbb{R}), \forall i = 1, \dots, r$ and $g_{r+1, r+1} \in \mathrm{Sp}_{2(n-r)}(\mathbb{R})$. For $i = 1, \dots, r$, because $g_{ii} \in \mathrm{Sp}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R})$ means that

$$a_i d_i - b_i c_i = 1.$$

From 2.10 and 2 we have

$$a_i^2 + c_i^2 = 1,$$

i.e. $g_{ii} \in \mathbf{S}^1$, for $i = 1, \dots, r$. We conclude therefore that

$$g \in G_F \text{ iff and only if } g = \begin{pmatrix} [g_{11}] & & 0 \\ & \ddots & \\ & & [g_{r,r}] \\ 0 & & & [g_{r+1, r+1}] \end{pmatrix},$$

where $g_{ii} \in \mathbf{S}^1, i = 1, \dots, r$ and $g_{r+1, r+1} \in \mathrm{Sp}_{2(n-2)}(\mathbb{R})$. □

Corollary 2.4. *If the functional $F \in \mathfrak{g}^*$ is presented by a matrix of type*

$$F = \begin{pmatrix} \begin{bmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{bmatrix} & & 0 \\ & \ddots & \\ & & \begin{bmatrix} 0 & -\lambda_r \\ \lambda_r & 0 \end{bmatrix} \\ 0 & & & \begin{bmatrix} 0 & \dots \\ \vdots & \dots \end{bmatrix} \end{pmatrix},$$

where all $\lambda_i, i = 1, \dots, n$ are pairwise different, then its stabilizer is

$$G_F \cong (\mathbf{S}^1)^r \times \mathrm{Sp}_{2(n-r)}(\mathbb{R})$$

and the corresponding Lie algebra is

$$\mathfrak{g}_F \cong (\mathrm{Lie} \mathbf{S}^1) \times (\mathrm{Lie} \mathbf{S}^1) \times \dots \times (\mathrm{Lie} \mathbf{S}^1) \times \mathfrak{sp}_{2(n-r)}(\mathbb{R})$$

$$\cong \mathbb{R}^r \times \mathfrak{sp}_{2(n-r)}(\mathbb{R}).$$

3. CONSTRUCTION OF DEGENERATE PRINCIPAL SERIES REPRESENTATIONS

We give in this section a geometric realization of degenerate principal series representations by multidimensional quantization procedure and the modified versions. We use the root theory to construct polarization associated to the orbits.

Let us recall from [2]-[3] an important notion of polarization.

Definition 3.1. A triple $(\mathfrak{p}, \rho, \sigma_0)$ is a $(\tilde{\sigma}, F)$ -polarization iff:

1. \mathfrak{p} is a complex Lie subalgebra of $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, containing $(\mathfrak{g}_F)_{\mathbb{C}}$.
2. The subalgebra \mathfrak{p} is invariant with respect to all the operators $\text{Ad}_{\mathfrak{g}_{\mathbb{C}}} x, x \in G_F$.
3. The vector space $\mathfrak{p} + \bar{\mathfrak{p}}$ is the complexification of the real Lie subalgebra $\mathfrak{m} = (\mathfrak{p} + \bar{\mathfrak{p}}) \cap \mathfrak{g}$.
4. All the subgroups M_0, H_0, M, H are closed in G , where M_0 (resp. H_0) is the connected subgroup of G , corresponding to the Lie algebra \mathfrak{m} (resp. $\mathfrak{h} = \mathfrak{p} \cap \mathfrak{g}$) and $M := G_F \ltimes M_0, H := G_F \ltimes H_0$.
5. σ_0 is an irreducible representation of the group H_0 in a Hilbert space V such that: (i) the restriction $\sigma|_{G_F \cap H_0}$ is a multiple of the restriction to $G_F \cap H_0$ of $\tilde{\sigma}\chi_F$ and (ii) the point σ_0 is fixed under the action of group G_F in the dual $\widehat{H_0}$ of the group H_0 .
6. ρ is a representation of the complex algebra \mathfrak{p} in V , which satisfies all the Nelson's conditions for H_0 , and $\rho|_{\mathfrak{h}} = D\sigma_0$.

Remark 3.2. For the group $G = \text{Sp}_{2n}(\mathbb{R})$ and the functional F of special type as in the previous section, the stabilizer \mathfrak{g}_F contains a Cartan subalgebra. It is therefore naturally to choose the polarizing complex subalgebras between parabolic ones. Let us denote by A the split torus of that cartan subgroup. It is easy to see that the centralizer $\mathcal{Z}(A)$ is reductive and is coincided with the stabilizer of the functional F . Denote $\mathcal{Z}(A) = MA$ the Cartan-Levi-Maltsev decomposition into the product of Abelian and semi-simple parts, where $A = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ and $M = \text{Sp}_{2(n-r)}(\mathbb{R})$.

Lemma 3.3. *The two-fold covering of the stabilizer $G_F = MA$ is $G_F^{\mathbb{Z}/(2)} = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \times \text{Mp}_{2(n-r)}(\mathbb{R})$*

Theorem 3.4. *Let $P = MAN$ be Langlands decomposition of a parabolic subgroup P with Lie algebra $\mathfrak{p} = \text{Lie } P$, $\mathfrak{p}_{\mathbb{C}}$ its complexification, $\sigma \in \hat{M}_{disc}$ a discrete series irreducible representation of $M = \text{Sp}_{2(n-r)}(\mathbb{R})$ and $\chi_F = \exp(\frac{2\pi i}{h}\langle F, \cdot \rangle)$ to be the character of G_F , extended by null from A . Then $(\mathfrak{p}_{\mathbb{C}}, P, F, \sigma\chi_F)$ is a (σ, F) polarization. The multidimensional quantization procedure gives us the degenerate principal series representations.*

Proof. The first assertion is just followed from the definition of (P, N) pairs, which are constructed from the theory of root of parabolic pairs.

Let us recall about the root theory for $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{R})$ with respect to the pair $(G, A) = (\mathrm{Sp}_{2n}(\mathbb{R}), \mathbb{S}^1 \times \cdots \times \mathbb{S}^1)$ of a split torus A . Consider the complexification $\mathfrak{g}_{\mathbb{C}}$. If α is a functional over a subalgebra $\mathfrak{a}_{\mathbb{C}} \cong \mathbb{C}^r$, then the corresponding root space

$$\mathfrak{g}^{\alpha} := \{X \in \mathfrak{g}_{\mathbb{C}} \ ; \ [H, X] = \alpha(H)X, \forall H \in \mathfrak{a}_{\mathbb{C}}\}.$$

If α and \mathfrak{g}^{α} are non-zero, then α is called a root. In that case, $\dim_{\mathbb{C}} \mathfrak{g}^{\alpha} = 1$. Denote the unique H_{α} such that $\alpha(H_{\alpha}) = 2$, the unique X_{α} and $X_{-\alpha}$, such that

$$[H_{\alpha}, X_{\alpha}] = 2X_{\alpha},$$

$$[H_{\alpha}, X_{-\alpha}] = -2X_{-\alpha}.$$

The triple $(H_{\alpha}, X_{\alpha}, X_{-\alpha})$ form a complex Lie subalgebra, which is isomorphic to the complexification of either $\mathfrak{sl}_2(\mathbb{R})$ or \mathfrak{su}_2 . In the first case, we call the root *non-compact* and in the second - *compact*. One denotes the set of all roots by $\Delta(\mathfrak{g}, \mathfrak{a})$ and calls it the *root system* with respect to the split torus A . Denote also the set of all compact (respectively, noncompact) roots by $\Delta_c(\mathfrak{g}, \mathfrak{a})$ (resp., $\Delta_n(\mathfrak{g}, \mathfrak{a})$). Let us denote by

$$\rho(\Delta_c^+(\mathfrak{g}, \mathfrak{a})) := \frac{1}{2} \sum_{\alpha \in \Delta_c^+(\mathfrak{g}, \mathfrak{a})} \alpha \quad (\text{resp., } \rho(\Delta_n^+(\mathfrak{g}, \mathfrak{a})) := \frac{1}{2} \sum_{\alpha \in \Delta_n^+(\mathfrak{g}, \mathfrak{a})} \alpha$$

) the half-sum of all compact (resp., noncompact) roots.

Remark that in our case, $\mathfrak{g}^0 = (\mathfrak{g}_F)_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{sp}_{2(n-r)}(\mathbb{C})$ and we have

$$\mathfrak{g}_{\mathbb{C}} = (\mathfrak{g}_F)_{\mathbb{C}} \oplus \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}^{\alpha}.$$

With each root α one associates a reflection R_{α} in the space $\mathfrak{a}_{\mathbb{C}}^*$

$$R_{\alpha}(\beta) := \beta - 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

The Weyl group $W(G, A)$ is generated by all these reflection and one can fix one of the *fundamental domain (camera)* to define the cones of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{a})$ and the corresponding cones of positive compact roots $\Delta_c^+(\mathfrak{g}, \mathfrak{a})$ and noncompact roots $\Delta_n^+(\mathfrak{g}, \mathfrak{a})$.

Choose

$$D\delta^F := \rho(\Delta_n^+(\mathfrak{g}, \mathfrak{a})) - \rho(\Delta_c^+(\mathfrak{g}, \mathfrak{a})).$$

Because in our case we can choose a charcters δ^F of the two-fold covering $G_F^{\mathbb{Z}/(2)} = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \times \mathrm{Mp}_{2(n-r)}(\mathbb{R})$ we can choose a complex parabolic subalgebra

$$\mathfrak{p} := (\mathfrak{g}_F)_{\mathbb{C}} \oplus \sum_{\alpha \in \Delta_n^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}^{\alpha} \oplus \sum_{\alpha \in \Delta_c^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}^{-\alpha}$$

From the properties of parabolic subgroups, it is easy to check that we have a (σ, F) polarization.

Let us first explain the construction of the degenerate principal series. We recall some notations from D. Vogan : By $R(G_F)$ denote the set of all the so called G_F -regular unitary pseudo-characters (Λ, F) consisting of a \mathfrak{g}_F -regular functional $F \in \mathfrak{g}_F^*$ and a unitary representation Λ of G_F with differential

$$D\Lambda = \left(\frac{2\pi i}{h} F + D\delta^F \right) \text{Id} \quad .$$

Remark that the last condition is equivalent to the assertion that

$$\Lambda|_{(G_F)_0} = \text{mult } \delta^F \chi_F \quad ,$$

what figures in the orbit method.

Denote by $R^{irr}(G_F)$ the subset of $R(G_F)$, consisting of the irreducible pseudo-characters. For a fixed $F \in \mathfrak{g}_F^*$, denote

$$R(G_F, F) := \{(\Lambda, F) \in R(G_F); D\Lambda = \left(\frac{2\pi i}{h} F + \delta^F \right) \text{Id}\}$$

and

$$R^{irr}(G_F, F) := R(G_F, F) \cap R^{irr}(G_F).$$

The Weyl group

$$W(G, A) := \mathcal{N}_G(A)/A$$

acts on both $R(G_F)$ and $R^{irr}(G_F)$.

Recall *Harish-Chandra construction of $\pi(\Lambda, F)$* : Consider the characters of type

$$\xi_\alpha(\cdot) := \exp \langle \alpha, \cdot \rangle$$

for each $\alpha \in \Delta$. Let us denote

$$\mathbf{F} := \{x \in G_F; x \text{ centralizers } \mathfrak{m} \text{ and } |\xi_\alpha| = 1, \forall \alpha \in \Delta\}.$$

Then

$$G_F := \mathbf{F}(G_F)_0 := \mathbf{F} \ltimes (G_F)_0,$$

$$G_F \cap G_0 = (G_F)_0$$

and

$$\mathbf{F}(G_F)_0 = \mathbf{F} \ltimes (G_F)_0.$$

Denote

$$\mathfrak{k}_{\mathfrak{m}} = ((\mathfrak{g}_F)_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{m},c}} \mathfrak{g}^\alpha) \cap \mathfrak{g}$$

and K_{M_0} the corresponding analytic subgroup. Let us denote $\pi^{M_0}(F)$ the irreducible unitary representation of M_0 , which is square-integrable modulo the center of M_0 , and which is associated with F . This representation, following Harish-Chandra is characterized by the following condition.

The restriction $\pi^{G_0}(F)|_{K_{M_0}}$ contains the (finite dimensional) irreducible unitary representation of K_{M_0} with the dominant weight $\frac{2\pi i}{h} F + D\delta^F$ with respect to $\Delta_{\mathfrak{m},c}^+$, as a minimal K_{M_0} -type

Now a representation $\pi^{\mathbf{F}M_0}(\Lambda, F)$ of $\mathbf{F}M_0 := \mathbf{F} \ltimes M_0$ can be constructed as follows

$$\pi^{\mathbf{F}M_0}(\Lambda, F)(y.x) := \Lambda(y) \otimes \pi^{M_0}(F)(x), \forall x \in M_0, y \in \mathbf{F}.$$

Let $P = MN$ be a parabolic subgroup of G with the Levi component M and the unipotent radical N ,

$$\begin{aligned} M &= \mathbf{F}M_0 = \mathbf{F}M_0, \\ P &= MN = (\mathbf{F}M_0) \ltimes N. \end{aligned}$$

Define now

$$\pi(\Lambda, F) := \text{Ind}_{\mathbf{F}M_0 \ltimes N}^G (\pi^{\mathbf{F}M_0} \otimes \text{Id}_N).$$

Recall that if Λ is irreducible and F is \mathfrak{g}_F -regular, the representation $\pi(\Lambda, \lambda)$ is irreducible.

Now we have

$$\Lambda \in R^{irr}(G_F, F) = X_G^{irr}(F) \xrightarrow{1-1} \tilde{X}^{irr}(F).$$

This means that there exists a unique $\tau = \sigma\chi_F \in \widehat{G_F}$, with $\sigma \in \hat{M}_{disc}$, such that

$$\tau|_{P_0} = \text{mult}(\chi_F \delta^F)$$

and $\Lambda = \tau\delta^F$,

$$\pi(\tau\delta^F, F) = \text{Ind}(G; \mathfrak{p}_{\mathbb{C}}, P, \tau\chi_F).$$

□

Remark 3.5. The restrictions of F to the radical rG_F and unipotent radical uG_F are equal to the split torus A , with Lie algebra $\mathfrak{a} \cong \mathbb{R}^r$ and therefore the reductions to the radical and unipotent radical of P give the same results as the above exposed ones.

Lemma 3.6. *The $U(1)$ -covering of the stabilizer $G_F = MA$ is $G_F^{U(1)} = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \times \text{Mp}_{2(n-r)}^c(\mathbb{R})$*

Corollary 3.7. *Lifting to the $\mathbb{Z}/(2)$ - and $U(1)$ - coverings we obtain the Shale-Weil representations of the symplectic groups $\text{Sp}_{2n}(\mathbb{R})$.*

Remark 3.8. Up to conjugation, the maximal parabolic subgroups are of the form

$$\begin{pmatrix} \cos \theta & 0 & \cdots & 0 & 0 & \cdots & 0 & -\sin \theta \\ 0 & & & & & & & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & & & \text{Id} & & & 0 \\ 0 & & & & & & & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & & & & & & 0 \\ \sin \theta & 0 & \cdots & 0 & 0 & \cdots & 0 & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} 1 & x_1 & \cdots & x_n & y_1 & \cdots & y_n & z \\ 0 & & & & & & & -y_1 \\ \vdots & & & & & & & \vdots \\ 0 & & & & & & & -y_n \\ 0 & & & & \text{Sp}_{2(n-1)} & & & x_1 \\ \vdots & & & & & & & \vdots \\ 0 & & & & & & & x_n \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and therefore the construction gives us the representation of degenerate principal series considered by S. T. Lee in [6].

Corollary 3.9. *For the maximal parabolic subgroups $P = (\mathbb{S}^1 \times \mathrm{Sp}_{2(n-1)}(\mathbb{R})) \ltimes (\mathbb{R}^{2n} \times \mathbb{R})$ the corresponding degenerate principal series representations can be realized in the space of homogeneous functions of $2n$ variables.*

ACKNOWLEDGMENTS

This work was completed during the stay of the first author as a visiting mathematician at the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. He would like to thank the Abdus Salam ICTP for the hospitality, without which this work would not have been possible and the IMU Commission of Development and Exchange for a generous grant.

This work is supported in part by the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, the IMU Commission on Development and Exchange and the Vietnam National Foundation for Research in Fundamental Sciences.

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